A REMARK ON HOLOMORPHIC SECTIONS OF CERTAIN HOLOMORPHIC FAMILIES OF RIEMANN SURFACES

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Abstract. In this paper, we study a special holomorphic family (M, π, R) of closed Riemann surfaces of genus two over a fourth punctured torus R, which is a kind of a Kodaira surface and is constructed by Riera. We give two explicit defining equations for (M, π, R) by using elliptic functions, and determine all the holomorphic sections of (M, π, R). Proofs will appear elsewhere.

1. Introduction

Let us consider a Diophantine equation

$$\sum_{i+j+k=N} A_{ijk} X^i Y^j Z^k = 0 \tag{1}$$

over the function field K of a closed Riemann surgace $\hat{R}$. Here K is the field of all meromorphic functions on $\hat{R}$ and the coefficients A_{ijk} are elements of K. The problem is to find the solutions $(X, Y, K) \in P^2(K)$ of the function equation (1) over $\hat{R}$.

This problem is reformulated geometrically in the following way. It is assumed that we find a Zariski open subset R of $\hat{R}$ and a Zariski open subset M of the algebraic surface $\hat{M}$ defined by

$$\hat{M} = \{([x, y, z], t) \in P^2(\mathbf{C}) \times \hat{R} \mid \sum_{i+j+k=N} A_{ijk}(t) x^i y^j z^k = 0\}$$

such that the holomorphic map $\pi\colon M \to R$ given by $\pi([x, y, z], t) = t$ satisfies the two conditions:

1. (1) π is of maximal rank at every point of M, and
2. (2) for every $t \in R$, the fiber $S_t = \pi^{-1}(t)$ of M over t is a Riemann surface of fixed finite type (g, n), where g is the genus of S_t and n is the number of punctures of S_t.

We call such a triplet (M, π, R) is a *holomorphic family of Riemann surfaces* of type (g, n) over R.

We assume throughout this paper that a holomorphic family (M, π, R) of Riemann surfaces is of type (g, n) with $2g - 2 + n > 0$ and the base surface R is of finte type, i.e., a Riemann surface obtained by removing at most a finite number of points from a closed Riemann surface.

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Two holomorphic families (M_1, π_1, R) and (M_2, π_2, R) of Riemann surfaces are called *isomorphic* if there exists a biholomorphic map $f\colon M_1 \to M_2$ with $\pi_1 = \pi_2 \circ f$. A holomorphic family (M, π, R) of Riemann surfaces is *locally trivial* if for every point $p_0 \in R$ there exsits a neighborhood U_0 of p_0 in R such that $(\pi^{-1}(U_0), \pi|\pi^{-1}(U_0), U_0)$ is isomorphic to the trivial holomorphic family $(U_0 \times S_{p_0}, \pi_0, U_0)$, where $\pi_0\colon U_0 \times S_{p_0} \to U_0$ is the canonical projection. It is known that a holomorphic family (M, π, R) of Riemann surfaces is locally trivial if and only if the fibers S_t are all isomorphic.

A holomorphic map $s\colon R \to M$ is said to be a *holomorphic section* of a holomorphic family (M, π, R) of Riemann surfaces if s stisfies $\pi \circ s = \mathrm{id}$ on R. Note that if $(X, Y, K) \in P^2(K)$ satisfies the function equation (1), then $s(t) = ([X(t), Y[t], Z[t], t)$ gives rise to a holomorphic section of (M, π, R).

By using theory of Teichmüller space, Imayoshi and Shiga in [6] gave a proof for a finiteness theorem of sections (Mordell conjecture) and a finiteness theorem of families (Shafarevich conjecture). See also Arakelov [1], Faltings [3], Grauert [4], Jost and Yau [7], Manin [9], McMullen [10], and Parshin [11].

In this paper we deal with a special holomorphic family (M, π, R) of closed Riemann surfaces of genus two over a fourth punctured torus R, which is a kind of a Kodaira surface as [8] and is constructed by Riera in [12]. We give two explicit defining equations for (M, π, R) by using elliptic functions, and determine all the holomorphic sections of (M, π, R).

2. Construction of a certain holomorphic family (M, π, R) of Riemann surfaces

Now we explain briefly a construction of our holomorphic family (M, π, R) of Riemann surfaces, which is due to Riera [12].

Take a point τ in the upper half-plane $\mathbf{H}$ in the complex plane $\mathbf{C}$. Let $\Gamma_{1,\tau}$ be the discrete subgroup of $\mathrm{Aut}(\mathbf{C})$ generated by two translations $z \mapsto z + 1$ and $z \mapsto z + \tau$. Denote by $\hat{T}$ a torus defiend by the quotient space $\mathbf{C}/\Gamma_{1,\tau} = \{[z] \mid z \in \mathbf{C}\}$. We set $p_0 = [0] \in \hat{T}$ and $T = \hat{T} \setminus \{p_0\}$.

For a point $p \in T$ we take two replicas of the torus $\hat{T}$ cut along a simple arc from p to p_0, and call them sheet I and sheet II. The cut on each sheet has two edges, labeled $+$ edge and $-$ edge. To construct a Riemann surface X_p, we attach the $+$ edge on sheet I and the $-$ edge on sheet II, and then attach the $+$ edge on sheet II and the $-$ edge on sheet I. Then we obtain a closed Riemann surface X_p of genus two and the two-sheeted covering $X_p \to \hat{T}$ which is branched over p_0 and p with branch order 2. It should be noted that the above procedure depends, of course, not only on the choice of the point p but also on the choice of the "cut" from p to p_0. Essentially we can take four different "cuts" $\alpha_1, \alpha_2, \alpha_3$, and α_4 between p and p_0 (see Fig.1).

To specify the "cut" we construct a four-sheeted unbranched covering

$$\rho\colon R \to T \tag{2}$$

of T such that R is a torus with four punctures as follows: Let $\Gamma_{2,2\tau}$ be the discrete subgroup of $\mathrm{Aut}(\mathbf{C})$ generated by two translations $z \mapsto z + 2$ and $z \mapsto z + 2\tau$. Denote by $\hat{R}$ a torus defiend by the quotient space $\mathbf{C}/\Gamma_{2,2\tau} = \{[z] \mid z \in \mathbf{C}\}$. Let $\hat{\rho}\colon \hat{R} \to \hat{T}$ be the canonical projection given by $\hat{\rho}([z]) = [z]$. We set $R = \hat{\rho}^{-1}(T)$ and $\rho = \hat{\rho}|R$. The good thing is that a point $t = [z] \in R$ determines a point $p = \rho([z]) \in T$ and a "cut"

$\alpha = \hat{\rho}(\beta)$ from p to $p_0 = [0]$, where β is a simple arc on $\hat{R}$ from $[0]$ to t. Denote by S_t the closed Riemann surface of genus two which is a two-sheeted branched covering surface of $\hat{T}$ constructed by a "cut" $\alpha = \hat{\rho}(\beta)$. Note that the two-sheeted branched covering $\Pi_t \colon S_t \to \hat{T}$ is uniquely determined by the choice of $t \in R$ and does not depend on β.

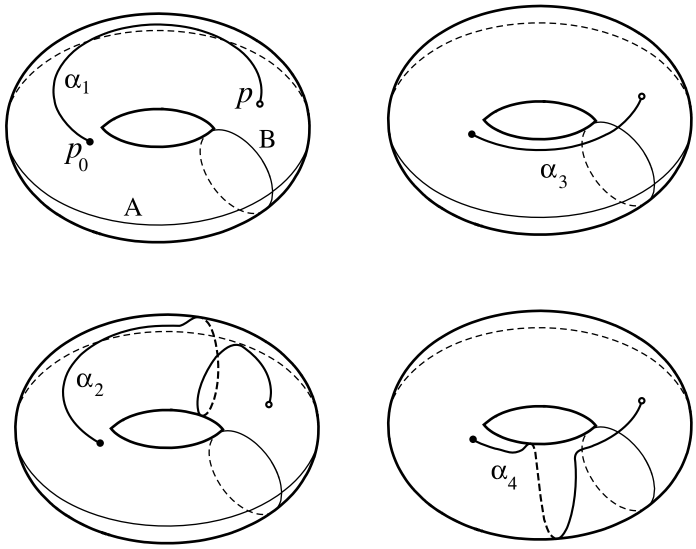

FIGURE 1. For each $p \in T$, there are precisely four two-sheeted branched coverings $S_{t_i} \to \hat{T}$, where $t_i \in R$ with $\hat{\rho}(t_i) = p$ and the cut α_i is given by the projection $\hat{\rho}(\beta_i)$ of a simple arc β_i between $[0]$ and t_i on $\hat{R}$.

Now we have the following theorem (see Riera [12]):

Theorem 1. *In the above situation, let*

$$M = \bigsqcup_{t \in R} \{t\} \times S_t,$$
$$\pi \colon M \to R,\ \pi(t, q) = t.$$

Then (M, π, R) is a holomorphic family of closed Riemann surfaces of genus two over a fourth punctured torus R.

3. Defining equations for (M, π, R)

In this section we find defining equations for (M, π, R). For any point $t = [\tilde{t}\,] \in R$, Abel's theorem shows there exists a meromorphic function f_t on $\hat{T}$ which has two zeros $[0]$ and $\rho(t)$ of order one, and a pole $q_t = \rho(t)/2$ of order two. Moreover in order to determine f_t uniquely, we assume that $(df_t/dz)([0]) = 1$. This function f_t is given explicitly as follows (see Clemens [2]):

$$f_t([z]) = \frac{1}{\theta'(1/2+\tau/2)} \times \frac{\theta\left(-\tilde{t}/2+1/2+\tau/2\right)^2}{\theta\left(-\tilde{t}+1/2+\tau/2\right)} \times \frac{\theta(z+1/2+\tau/2)\,\theta\left(z-\tilde{t}+1/2+\tau/2\right)}{\theta\left(z-\tilde{t}/2+1/2+\tau/2\right)^2}. \tag{3}$$

Here the theta function $\theta(z,\tau)$ is defined by

$$\theta(z,\tau) = \sum_{k=-\infty}^{\infty} e^{\pi i(k^2\tau+2kz)}, \quad z \in \mathbf{C}.$$

Then we have the following assertion:

Theorem 2. *In the above situation, let*

$$M_E = \{(t,p,w) \in R \times \hat{T} \times \hat{\mathbf{C}} \mid w^2 = f_t(p)\},$$
$$\pi_E : M_E \to R,\ \pi_E(t,p,w) = t.$$

Then the triplet (M_E, π_E, R) is a holomorphic family of closed Riemann surfaces of genus two, and it is isomorphic to $(\mathcal{M}, \pi, R)$ in Theorem 1.

We find another defining equation for (M, π, R). The holomorophic map $f_t\colon \hat{T} \to \hat{\mathbf{C}}$ has four branch points q_t (pole), $a(t), b(t)$, and $c(t)$, where

$$a(t) = f_t([(\tilde{t}+1)/2]),$$
$$b(t) = f_t([(\tilde{t}+\tau)/2]),$$
$$c(t) = f_t([(\tilde{t}+1+\tau)/2]).$$

Let g_t be the meromorphic function on $\hat{T}$ of degree 3 satisfying

(1) g_t has simple zeros $[(\tilde{t}+1)/2], [(\tilde{t}+\tau)/2], [(\tilde{t}+1+\tau)/2]$,
(2) g_t has a pole $[\tilde{t}]$ of order 3, and
(3) $g_t([0]) = i$.

This function g_t is given by

$$g_t(z) = ie^{-2\pi iz} \times \frac{\theta\left(-\tilde{t}/2+1/2+\tau/2\right)^3}{\theta\left(-\tilde{t}/2\right)\,\theta\left(-\tilde{t}/2+1/2\right)\,\theta\left(-\tilde{t}/2+\tau/2\right)} \times \frac{\theta\left(z-\tilde{t}/2\right)\,\theta\left(z-\tilde{t}/2+1/2\right)\,\theta\left(z-\tilde{t}/2+\tau/2\right)}{\theta\left(z-\tilde{t}/2+1/2+\tau/2\right)^3}.$$

Setting $x = f_t, y = g_t$, we have a functional relation

$$y^2 = \frac{1}{a(t)b(t)c(t)}\,(x-a(t))\,(x-b(t))\,(x-c(t)) \tag{4}$$

on $\hat{T}$.

Now we have the following theorem:

Theorem 3. *In the above situation, let*

$$P_t(x) = (x^2 - a(t))(x^2 - b(t))(x^2 - c(t)),$$
$$M_{HE} = \{(t, x, y) \in R \times \hat{\mathbf{C}} \times \hat{\mathbf{C}} \mid y^2 = P_t(x)\},$$
$$\pi_{HE} \colon M_{HE} \to R,\ \pi_{HE}(t, x, y) = t.$$

Then the triplet (M_{HE}, π_{HE}, R) *is a holomorphic family of closed Riemann surfaces of genus two, and it is isomorphic to* (M, π, R) *in Theorem 1.*

4. Holomorphci sections (M, π, R)

Let us study the holomorphic sections of (M, π, R) in Theorem 1. The following is our main theorem in this paper.

Theorem 4. *The holomorphic family* (M, π, R) *of closed Riemann surfaces of genus two in Theorem 1 has exactly two holomorphic sections* s_1, s_2. *These sections are given by* $s_1(t) = (t, p_0)$ *and* $s_2(t) = (t, \rho(t))$ *for every* $t \in R$.

In order to prove this theorem, we need the following two theorems (cf. Imayoshi [5], Theorem 4 and Theorem 5):

Theorem 5. *The holomorphic family* (M, π, R) *in Theorem 1 has a canonical completion* $(\hat{M}, \hat{\pi}, \hat{R})$, *where* $\hat{M}$ *is a compact two dimensional normal complex analytic space and* $\hat{\pi} \colon \hat{M} \to \hat{R}$ *is holomorphic. Moreover every holomorphic section* $s \colon R \to M$ *has a holomorphic extension* $\hat{s} \colon \hat{R} \to \hat{M}$.

Theorem 6. *The holomorphic map* $\Pi \colon M = \bigsqcup_{t \in R} \{t\} \times S_t \to \hat{T}$ *defined by* $\Pi(t, q) = \Pi_t(q)$ *has a holomorphic extension* $\hat{\Pi} \colon \hat{M} \to \hat{T}$.

Theorem 6 is proved by Theorems 2, 3, and 5.

Now we can prove Theorem 4 as follows: Let $s \colon R \to M$ be an arbitrary holomorphic section of (M, π, R). Theorem 5 and 6 imply that the holomorphic map $\varphi = \Pi \circ s \colon R \to \hat{T}$ has a holomorphic extention $\hat{\varphi} = \hat{\Pi} \circ \hat{s} \colon \hat{R} \to \hat{T}$. Let $\tilde{\varphi} \colon \mathbf{C} \to \mathbf{C}$ is a lift of $\hat{\varphi} \colon \hat{R} \to \hat{T}$. Then $\tilde{\varphi}(z) = Az + B, z \in \mathbf{C}$ for some constants $A, B \in \mathbf{C}$. Since $\varphi = \Pi \circ s$, we can show that we may assume that $A = 0, B = 0$, or $A = 1, B = 0$. In the case $A = 0, B = 0$, we have the section $s_1(t) = (t, p_0)$, and in the case $A = 1, B = 0$, we have the section $s_2(t) = (t, \rho[t])$.

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